Quantum Metrology with Open Dynamical Systems

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This Letter studies quantum limits to dynamical sensors in the presence of decoherence. A modified purification approach is used to obtain tighter quantum detection and estimation error bounds for optomechanical force sensing and optical phase sensing. When optical loss is present, these bounds are found to obey shot-noise scalings for arbitrary quantum states of light under certain realistic conditions, thus ruling out the possibility of asymptotic Heisenberg error scalings with respect to the average photon flux under those conditions. The proposed bounds are expected to be approachable using current quantum optics technology.

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The laws of quantum mechanics impose fundamental limitations to the accuracy of measurements, and a fundamental question in quantum measurement theory is how such limitations affect precision sensing applications, such as gravitational-wave detection, optical interferometry, and atomic magnetometry and gyroscopy [1]. With the rapid recent advance in quantum optomechanics [2] and atomic [3] technologies, quantum sensing limits have received renewed interest and are expected to play a key role in future precision measurement applications.

Many realistic sensors, such as gravitational-wave detectors, perform continuous measurements of time-varying signals (commonly called waveforms). For such sensors, a quantum Cramér-Rao bound (QCRB) for waveform estimation [4] and a quantum fidelity bound for waveform detection [5] have recently been proved, generalizing earlier seminal results by Helstrom [6]. These bounds are not expected to be tight when decoherence is significant, however, as Refs. [4, 5] use a purification approach that does not account for the inaccessibility of the environment. Given the ubiquity of decoherence in quantum experiments, the relevance of the bounds to practical situations may be questioned.

One way to account for decoherence is to employ the concepts of mixed states, effects, and operations [7]. Such an approach has been successful in the study of single-parameter estimation problems [8, 9], but becomes intractable for nontrivial quantum dynamics. To retain the convenience of a larger Hilbert space, here I extend a modified purification approach due to Escher *et al.* [9] and apply it to more general open-system detection and estimation problems beyond the paradigm of single-parameter estimation considered by previous work [8, 9]. In particular, I show that

1. A quantum model of optomechanical force sensing can be transformed to an optical phase sensing problem with classical phase shift, such that a unified formalism can treat both problems and produce tighter bounds than the results in Refs. [4, 5].

2. For optical phase waveform detection with loss and vacuum noise, the errors obey lower bounds that scale with the average photon flux akin to reduced shot-noise limits, provided that the phase shift or the quantum efficiency is small enough (the precise conditions will be given later). This rules out Heisenberg scaling of the detectable phase shift [10] in the high-flux limit under such conditions, as well as any significant enhancement of the error exponent by quantum illumination [11] in the low-efficiency limit.

3. The mean-square error for lossy optical phase waveform estimation also observes a limit with shot-noise scaling, which generalizes the single-parameter results in Refs. [8, 9] and rules out the kind of quantum-enhanced scalings suggested by Refs. [12] in the high-flux limit.

These results not only provide more general and realistic quantum limits that can be approached using current quantum optics technology [13, 14], but may also be relevant to more general studies of quantum metrology and quantum information, such as quantum speed limits [15] and quantum phase transitions [16].

Let x be the vector of unknown parameters to be estimated and y be the observation. Within the purification approach [4, 5], the dynamics of a quantum sensor is modeled by unitary evolution (U_x as a function of x) of an initial pure density operator $\rho = |\Psi\rangle\langle\Psi|$, and measurements are modeled by a final-time positive operator-valued measure (POVM) $E(y)$ using the principle of deferred measurement [17]. The likelihood function becomes $P(y|x) = \mathrm{tr}\left[E(y)U_x\rho U_x^\dagger\right]$. For continous-time problems, discrete time is first assumed and the continuous limit is taken at the end of calculations. Refs. [4, 5] derive quantum bounds by considering the density operator $U_x\rho U_x^\dagger$.

Suppose that the Hilbert space ($\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$) is divided into an accessible part ($\mathcal{H}_A$) and an inaccessi-

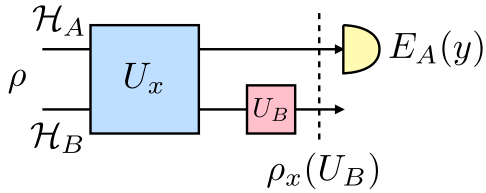

FIG. 1. Quantum circuit diagram [17] for the modified purification approach.

ble part ($\mathcal{H}_B$). The POVM should now be written as $E(y) = E_A(y) \otimes 1_B$, where $E_A(y)$ is a POVM on $\mathcal{H}_A$ and 1_B is the identity operator on $\mathcal{H}_B$, which accounts for the fact that $\mathcal{H}_B$ cannot be measured. The key to the modified purification approach, as illustrated by Fig. 1, is to recognize that the likelihood function is unchanged if any arbitrary x-dependent unitary U_B on $\mathcal{H}_B$ is applied before the POVM:

$$P(y|x) = \operatorname{tr}\left\{\left[E_A(y) \otimes 1_B\right] \rho_x(U_B)\right\}, \tag{1}$$

where

$$\rho_x(U_B) \equiv (1_A \otimes U_B^\dagger) U_x \rho U_x^\dagger (1_A \otimes U_B) \tag{2}$$

is a purification of $\operatorname{tr}_B(U_x \rho U_x^\dagger)$, such that $\operatorname{tr}_B \rho_x = \operatorname{tr}_B(U_x \rho U_x^\dagger)$. Judicious choices of U_B can result in tighter quantum bounds as a function of $\rho_x(U_B)$.

First, suppose that $x^{(0)}$ and $x^{(1)}$ are two hypotheses for x and $\tilde{x}(y)$ is the estimate. The following theorems are applications of the modified purification and Helstrom's bounds for pure states [6]:

Theorem 1 (Fidelity bound, Neyman-Pearson criterion)**.** *For any POVM measurement $E_A(y)$ of $\operatorname{tr}_B \rho_x$ in $\mathcal{H}_A$, the miss probability $P_{01} \equiv \int_{\tilde{x}=0} dy P(y|x=1)$ given a constraint on the false-alarm probability $P_{10} \equiv \int_{\tilde{x}=1} dy P(y|x=0) \leq \alpha$ satisfies*

$$\begin{aligned} P_{01} &\geq \beta(\alpha, F) \\ &\equiv \begin{cases} 1 - \left[\sqrt{\alpha F} + \sqrt{(1-\alpha)(1-F)}\right]^2, & \alpha < F, \\ 0, & \alpha \geq F, \end{cases} \end{aligned} \tag{3}$$

where F is the fidelity between the following pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\rho_0 \equiv U_0 |\Psi\rangle\langle\Psi| U_0^\dagger, \quad \rho_1 \equiv U_B^\dagger U_1 |\Psi\rangle\langle\Psi| U_1^\dagger U_B, \tag{4}$$

$$F(\rho_0, \rho_1) \equiv \left|\langle\Psi| U_1^\dagger U_B^\dagger U_0 |\Psi\rangle\right|^2, \tag{5}$$

$\rho_m \equiv \rho_{x^{(m)}}$, $U_m \equiv U_{x^{(m)}}$, and $1_A \otimes U_B$, abbreviated as U_B, is an arbitrary unitary on $\mathcal{H}_B$.

Theorem 2 (Fidelity bound, Bayes criterion)**.** *The average error probability $P_e \equiv P_{10} P_0 + P_{01} P_1$ with prior probabilities P_0 and $P_1 = 1 - P_0$ satisfies*

$$P_e \geq \frac{1}{2}\left(1 - \sqrt{1 - 4 P_0 P_1 F}\right) \geq P_0 P_1 F. \tag{6}$$

Proof of Theorem 1 and 2. Ref. [6] shows that the bounds with the likelihood function $P(y|x) = \operatorname{tr}[E(y)\rho_x]$ are valid for any POVM $E(y)$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, so they must also be valid with $P(y|x) = \operatorname{tr}_A[E_A(y) \operatorname{tr}_B \rho_x] = \operatorname{tr}\{[E_A(y) \otimes 1_B]\rho_x\}$ for any POVM $E_A(y)$ on $\mathcal{H}_A$. □

Since the lower bounds are valid for any U_B, U_B should be chosen to increase F and tighten the bounds. The maximum F becomes the Uhlmann fidelity between mixed states $\operatorname{tr}_B \rho_0$ and $\operatorname{tr}_B \rho_1$ [17, 18]. The bound on P_e obtained using this method is thus weaker than the Helstrom bound for the mixed states [6], although it can be shown that the error exponent for the Uhlmann-fidelity bound is within 3dB of the optimal value [19].

Next, consider the estimation of continuous parameters x with prior distribution $P(x)$. A lower error bound is given by the following:

Theorem 3 (Bayesian quantum Cramér-Rao bound)**.** *The error covariance matrix $\Sigma \equiv \mathbb{E}(\tilde{x}-x)(\tilde{x}-x)^\top = \int dy dx P(y|x) P(x) (\tilde{x}-x)(\tilde{x}-x)^\top$ satisfies a matrix inequality given by*

$$\Sigma \geq (J^{(Q)} + J^{(C)})^{-1}, \tag{7}$$

where

$$J_{jk}^{(Q)} = -4 \int dx P(x) \left.\frac{\partial^2 F(\rho_{x'}, \rho_x)}{\partial x_j' \partial x_k'}\right|_{x'=x}, \tag{8}$$

$$J_{jk}^{(C)} = \int dx P(x) \frac{\partial \ln P(x)}{\partial x_j} \frac{\partial \ln P(x)}{\partial x_k}. \tag{9}$$

Proof. See Ref. [4] for a proof and Refs. [20] for the relation between $J^{(Q)}$ and F. □

U_B should again be chosen to reduce the quantum Fisher information (QFI) $J^{(Q)}(U_B)$ and thus tighten the QCRB.

As an application of the general theory, consider the estimation of a force $x(t)$, $t \in [t_0, t_f]$, on a quantum moving mirror via continuous optical measurements. For simplicity, assume that any optical cavity dynamics can be adiabatically eliminated. Let (q, p) be the mechanical position and momentum operators, and $a(t)$ be the annihilation operator for the one-dimensional optical field, which obeys $[a(t), a^\dagger(t')] = \delta(t - t')$ [21]. Suppose

$$U_x = \mathcal{T} \exp\left[\frac{1}{i\hbar} \int_{t_0}^{t_f} dt H(t)\right], \tag{10}$$

with a Hamiltonian given by

$$H = H_B(q, p, x(t)) - 2\hbar M k q I(t), \tag{11}$$

where $\mathcal{T}$ is the time-ordering superoperator, H_B is the mechanical Hamiltonian, M is the effective number of optical reflections by the mirror, k is the optical wavenumber, and $I(t) \equiv a^\dagger(t) a(t)$ is the photon flux.

In an optomechanics experiment, the mechanical oscillator is measured only through the optical field, so one can let $\mathcal{H}_B$ be the mechanical Hilbert space. Assume

$$U_B(t_f) = \exp\left[\frac{1}{i\hbar}\int_{t_0}^{t_f} dt H_B(q,p,x(t))\right], \tag{12}$$

and let $U_I \equiv U_B^\dagger(t_f)U_x$, which can be calculated using the interaction picture. The result is

$$U_I = \mathcal{T}\exp\left[2iMk\int_{t_0}^{t_f} dt q_I(t)I(t)\right], \tag{13}$$

with $q_I(t) \equiv U_B^\dagger(t)qU_B(t)$ and $\rho_x = U_I\rho U_I^\dagger$. If the mechanical dynamics is linear, $q_I(t)$ can be expressed in terms of the mechanical impulse-response function $h(t,t')$ as $q_I(t) = q_0(q,p,t) + \int_{t_0}^{t_f} dt' h(t,t')x(t')$. Apart from the transient solution q_0, the mechanical Hilbert space has been removed from the model using U_B, and the problem has been transformed to a problem of optical phase sensing, with the phase given by $\phi(t) = 2Mk\int_{t_0}^{t_f} dt' h(t,t')x(t')$.

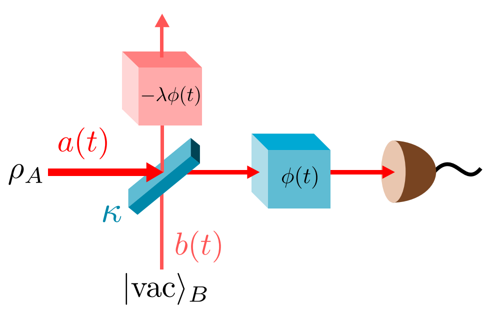

FIG. 2. A model of the lossy optical phase sensing problem.

I now turn to the problem of optical phase sensing, as depicted in Fig. 2. Consider first the detection problem. Let $\phi(t)$ be the phase shift between the two hypotheses, $U_0 = U_A U_{AB}$, and $U_1 = U_{AB}$, where

$$U_A = \exp\left[i\int_{t_0}^{t_f} dt\phi(t)I(t)\right], \tag{14}$$

$$U_{AB} = \exp\left\{i\kappa\int_{t_0}^{t_f} dt\left[a^\dagger(t)b(t) + a(t)b^\dagger(t)\right]\right\}. \tag{15}$$

U_{AB} models loss as a beam-splitter coupling with another optical field $b(t)$ in vacuum state $|0\rangle_B$ before the phase modulation, such that $|\Psi\rangle = |\psi\rangle_A \otimes |0\rangle_B$. U_{AB} can also account for loss after the modulation, as shown in Ref. [22]. The fidelity becomes

$$F = \left|\langle\Psi|U_{AB}^\dagger U_B^\dagger U_A U_{AB}|\Psi\rangle\right|^2. \tag{16}$$

With

$$U_B = \exp\left[i\int_{t_0}^{t_f} dt\lambda(t)\phi(t)b^\dagger(t)b(t)\right], \tag{17}$$

where $\lambda(t)$ is a free parameter to be specified later, Eq. (16) can be simplified using the $SU(2)$ disentangling theorem [23], as shown in Ref. [22]. The result is

$$F = \left|\langle\psi|\exp\left[\int_{t_0}^{t_f} dt I(t)\ln z(t)\right]|\psi\rangle\right|^2, \tag{18}$$

$$z(t) \equiv \eta e^{i\phi(t)} + (1-\eta)e^{-i\lambda(t)\phi(t)}, \tag{19}$$

with $\eta \equiv \cos^2\kappa$ defined as the quantum efficiency. For example, the coherent-state value is

$$F_{\mathrm{coh}} = \exp\left[-4\eta\int_{t_0}^{t_f} dt\langle I(t)\rangle\sin^2\frac{\phi(t)}{2}\right], \tag{20}$$

where $\langle O\rangle \equiv \langle\psi|O|\psi\rangle$. Measurements that can saturate the Bayes error bound in Theorem 2 for coherent states are known [6, 13].

To bound F in general, Jensen's inequality can be used if z is real and positive. The following lemma provides the necessary and sufficient condition:

Lemma 1. *There exists a λ such that $z \equiv \eta e^{i\phi} + (1-\eta)e^{-i\lambda\phi}$ is real and positive if and only if one of the following conditions is satisfied:*

$$(\mathrm{I}) : \eta < \frac{1}{2}, \tag{21}$$

$$(\mathrm{II}) : \eta \geq \frac{1}{2} \text{ and } |\sin\phi| \leq \frac{1-\eta}{\eta} \text{ and } \cos\phi > 0. \tag{22}$$

Proof. Consider the circle traced by $z(\lambda)$ centered at $\eta e^{i\phi}$ with radius $1-\eta$ on the complex plane. $z = |z| > 0$ for some λ is equivalent to the condition that the circle intersects the positive real axis, for which the necessary and sufficient condition is given by one of Eq. (21) (the circle encloses the origin for any ϕ and thus always intersects the axis) and Eq. (22) (the circle intersects the axis for some ϕ on the right-hand plane only). □

Eq. (22) holds when ϕ is sufficiently small. For example, $M \sim 100$, $|q| \sim 10^{-19}$ m, $2\pi/k \sim 1$ μm, and $(1-\eta)/\eta \sim 10^{-2}$ for LIGO [24], leading to $|\phi| \sim 2Mk|q| \sim 10^{-10}$, and Eq. (22) is easily satisfied.

The following theorem is a key technical result of this Letter:

Theorem 4. *If Eq. (21) or (22) is satisfied for all $\phi(t)$,*

$$F \geq \exp\left[\int_{t_0}^{t_f} dt\,\langle I(t)\rangle\ln|z(t)|^2\right] \equiv F_z. \tag{23}$$

Proof. With $z = |z| > 0$ under the condition in Lemma 1 and writing $|\psi\rangle$ as a superposition of eigenstates of $I(t)$, one can apply Jensen's inequality and get $\langle\exp(\int dtI\ln z)\rangle \geq \exp(\int dt\langle I\rangle\ln z)$. Eq. (23) then follows from Eq. (18). □

Compared with Eq. (20), F_z has the same shot-noise scaling with respect to the average photon flux $\langle I\rangle$, as $-\ln F_z$ scales linearly with $\langle I\rangle$. Since error-free detection with $F=0$ is possible with pure states [25], this shot-noise-scaling bound is a very strong result. It should also have implications for M-ary phase discrimination in general [26].

The following corollaries are some analytic consequences of Theorem 4 that exemplify its tightness:

Corollary 1. *If $|\phi| \ll 1$ and $|\eta\phi/(1-\eta)| \ll 1$,*

$$F_z \approx \exp\left[-\frac{\eta}{1-\eta}\int_{t_0}^{t_f} dt \langle I(t)\rangle \phi^2(t)\right]. \tag{24}$$

Proof. Let $\lambda_0 \equiv \eta/(1-\eta)$. Since $\operatorname{Im} z = \eta\sin\phi - (1-\eta)\sin\lambda\phi = 0$, $\lambda = \lambda_0[1+O(\phi^2)+O(\lambda_0^2\phi^2)]$, $|z|^2 = 1-4\eta(1-\eta)\sin^2[(1+\lambda)\phi/2] = 1-\lambda_0\phi^2[1+O(\phi^2)+O(\lambda_0^2\phi^2)]$, and $\ln|z|^2 = -\lambda_0\phi^2[1+O(\phi^2)+O(\lambda_0\phi^2)+O(\lambda_0^2\phi^2)]$, which determines the exponent of F_z in Eq. (23) and leads to Eq. (24). □

Eq. (24) differs from $F_{\mathrm{coh}} \approx \exp(-\eta\int dt\langle I\rangle\phi^2)$ by just a constant factor of $1/(1-\eta)$ in the exponent.

To obtain another measure of detection error, I formalize the concept of detectable phase shift [10] as follows:

Definition 1 (Detectable phase shift)**.** *A detectable phase shift $\phi'(t)$ given acceptable error probabilities α' and β' is a $\phi(t)$ that makes $P_{10} \le \alpha'$ and $P_{01} \le \beta'$.*

Corollary 2. *Under the conditions of Corollary 1,*

$$\int_{t_0}^{t_f} dt\langle I(t)\rangle\phi'^2(t) \gtrsim \frac{1-\eta}{\eta}(-\ln F'), \tag{25}$$

where $F' \equiv \max_{\beta'\ge\beta(\alpha',F)} F$ with β defined in Eq. (3).

Proof. Any achievable (P_{10}, P_{01}) must lie above the convex curve $P_{01} = \beta(P_{10}, F)$. This means that for $P_{10} \le \alpha'$ and $P_{01} \le \beta'$, $\beta' \ge \beta(\alpha', F)$ must hold, and hence $F \le \max_{\beta'\ge\beta(\alpha',F)} F \equiv F'(\alpha',\beta')$. Since $F \ge F_z$, $F_z \le F'$ must hold for the constraints on (P_{10}, P_{01}) to be possible. Eq. (25) then follows from Eq. (24). □

For example, if $\langle I(t)\rangle$ is constant,

$$\frac{1}{t_f - t_0}\int_{t_0}^{t_f} dt\phi'^2(t) \gtrsim \frac{1-\eta}{\eta N}(-\ln F'), \tag{26}$$

with $N \equiv (t_f - t_i)\langle I\rangle$. This is lower than the shot-noise limit by a constant factor of $1-\eta$ only, ruling out the kind of Heisenberg scaling suggested by Refs. [10] for lossy weak phase detection in the $N \to \infty$ limit.

Corollary 3. *If $\eta \ll 1$, $F_z \approx F_{\mathrm{coh}}$.*

Proof. Since $\operatorname{Im} z = 0$, $\lambda\phi = \sin^{-1}[\eta\sin\phi/(1-\eta)] = O(\eta)$, $z = \operatorname{Re} z = \eta\cos\phi + (1-\eta)\cos\lambda\phi = 1+\eta\cos\phi - \eta + O(\eta^2)$, and $\ln|z|^2 = -4\eta\sin^2(\phi/2) + O(\eta^2)$, which leads to $-\ln F_z = -[1+O(\eta)]\ln F_{\mathrm{coh}}$. □

Corollary 3 proves that, analogous to the case of target detection [27], the coherent state is near-optimal for any phase detection problem in the low-efficiency limit, ruling out any significant enhancement of the error exponent by quantum illumination [11] in this scenario. It remains an open question whether quantum illumination is useful for high-thermal-noise low-efficiency phase detection.

Consider now the estimation problem. The QFI $J^{(Q)}(t_j,t_k) \equiv \lim_{\delta t\to 0} J_{jk}^{(Q)}/\delta t^2$ for estimating $\phi(t)$ is found in Ref. [22] to be

$$\begin{aligned}\frac{J_\phi^{(Q)}(t,t')}{4} &= [\eta - \lambda(1-\eta)]^2 \langle \Delta I(t)\Delta I(t')\rangle \\ &\quad + (1+\lambda)^2\eta(1-\eta)\langle I(t)\rangle\delta(t-t'),\end{aligned} \tag{27}$$

where $\Delta I(t) \equiv I(t) - \langle I(t)\rangle$ and λ is assumed to be constant for simplicity. If the light source has stationary statistics, $\langle I(t)\rangle$ is constant, $\langle\Delta I(t)\Delta I(t')\rangle$ depends on $t-t'$ only, and a power spectral density $S_{\Delta I}(\omega)$ can be defined by $\langle\Delta I(t)\Delta I(t')\rangle = \int_{-\infty}^{\infty}(d\omega/2\pi)S_{\Delta I}(\omega)\exp[i\omega(t-t')]$. A spectral form of $J_\phi^{(Q)}$ is then $J_\phi^{(Q)}(\omega)/4 = [\eta-\lambda(1-\eta)]^2 S_{\Delta I}(\omega) + (1+\lambda)^2\eta(1-\eta)\langle I\rangle$. The minimum QFI with respect to λ is hence

$$\min_\lambda J_\phi^{(Q)}(\omega) = 4\left[\frac{1}{S_{\Delta I}(\omega)} + \frac{1-\eta}{\eta\langle I\rangle}\right]^{-1}. \tag{28}$$

This method of deriving an open-system QFI is the same as the one proposed in Refs. [9] and the result is a generalization of earlier ones for lossy static-phase estimation in Refs. [8, 9]. For $\phi(t) = \int_{-\infty}^{\infty} dt' g(t-t')x(t')$, the QCRB on $\Sigma_t \equiv \mathbb{E}[\tilde{x}(t) - x(t)]^2$ becomes [4]

$$\Sigma_t \ge \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{|g(\omega)|^2\min_\lambda J_\phi^{(Q)}(\omega) + J_x^{(C)}(\omega)}, \tag{29}$$

where $g(\omega)$ is the Fourier transform of $g(t)$ and $J_x^{(C)}(\omega)$ is the prior information in spectral form. Compared with the coherent-state value $J_{\phi,\mathrm{coh}}^{(Q)}(\omega) = 4\eta\langle I\rangle$, the QFI for any state is limited by the same shot-noise scaling ($\min_\lambda J_\phi^{(Q)}(\omega) \le 4\eta\langle I\rangle/(1-\eta)$). This rules out the kind of quantum-enhanced scalings suggested by Refs. [12] in the high-flux limit when loss is present.

For optomechanical force sensing, the detection and estimation error bounds here are more realistic to achieve than the bounds in Refs. [4, 5], as the former should be approachable using the quantum optics technology demonstrated in Refs. [13, 14], whereas the latter also requires measurement backaction noise to dominate the fluctuation of q and may require quantum noise cancellation [24, 28].

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Order of loss and phase modulation

Here I prove that the reduced state $\operatorname{tr}_B \rho_x$ is the same regardless of the order of the optical loss U_{AB} and the phase modulation U_A, viz.,

Lemma 2.

$$\operatorname{tr}_B \left(U_A U_{AB} \rho U_{AB}^\dagger U_A^\dagger\right) = \operatorname{tr}_B \left(U_{AB} U_A \rho U_A^\dagger U_{AB}^\dagger\right) \tag{30}$$

if $\rho = \rho_A \otimes \rho_B$ and ρ_B is a thermal state.

Proof. It suffices to prove it for one mode in $\mathcal{H}_A$ and one mode in $\mathcal{H}_B$; generalization to the multimode case is straightforward. Suppose

$$U_A = \exp(i\phi a^\dagger a), \qquad U_{AB} = \exp[i\kappa(a^\dagger b + ab^\dagger)], \qquad U_B = \exp(i\phi b^\dagger b), \tag{31}$$

where $[a, a^\dagger] = [b, b^\dagger] = 1$. Then

$$U_A U_{AB} U_A^\dagger = \exp[i\kappa(e^{i\phi} a^\dagger b + e^{-i\phi} ab^\dagger)] = U_B^\dagger U_{AB} U_B, \tag{32}$$

$$U_A U_{AB} = U_B^\dagger U_{AB} U_B U_A. \tag{33}$$

Since $U_B \rho_B U_B^\dagger = \rho_B$ for a thermal state,

$$\begin{aligned}
\mathrm{tr}_B \left(U_A U_{AB} \rho U_{AB}^\dagger U_A^\dagger \right) &= \mathrm{tr}_B \left(U_B^\dagger U_{AB} U_B U_A \rho U_A^\dagger U_B^\dagger U_{AB}^\dagger U_B \right) && (34)\\
&= \mathrm{tr}_B \left(U_B^\dagger U_{AB} U_A \rho U_A^\dagger U_{AB}^\dagger U_B \right) && (35)\\
&= \mathrm{tr}_B \left(U_{AB} U_A \rho U_A^\dagger U_{AB}^\dagger \right). && (36)
\end{aligned}$$

□

With the concatenation property of thermal-noise channels [29], any optical loss with thermal noise at any stage of a phase modulation experiment can be modeled by a single beam splitter before or after the modulation.

$SU(2)$ algebra

Consider again the single-mode case with

$$U_B = \exp(i\lambda\phi b^\dagger b). \tag{37}$$

First, compute the following quantity using the Heisenberg picture:

$$U_{AB}^\dagger U_B U_A U_{AB} = \exp(ig\phi), \qquad g = a'^\dagger a' - \lambda b'^\dagger b', \tag{38}$$

$$a' = \cos\kappa a + i\sin\kappa b, \qquad b' = \cos\kappa b + i\sin\kappa a. \tag{39}$$

This gives

$$g = \mu a^\dagger a + \nu b^\dagger b + i\gamma(a^\dagger b - ab^\dagger), \tag{40}$$

$$\mu \equiv \cos^2\kappa - \lambda\sin^2\kappa, \tag{41}$$

$$\nu \equiv \sin^2\kappa - \lambda\cos^2\kappa, \tag{42}$$

$$\gamma \equiv (1+\lambda)\sin\kappa\cos\kappa. \tag{43}$$

Next, define $SU(2)$ operators as

$$J_- \equiv a^\dagger b, \qquad J_+ \equiv ab^\dagger, \qquad J_3 \equiv \frac{1}{2}\left(b^\dagger b - a^\dagger a\right), \qquad J \equiv \frac{1}{2}\left(b^\dagger b + a^\dagger a\right), \tag{44}$$

where J commutes with the rest of the operators. In terms of the redefined operators,

$$\exp(ig\phi) = \exp\left[i\phi(\mu+\nu)J\right]\exp\left[i\phi(\nu-\mu)J_3 - \gamma\phi J_- + \gamma\phi J_+\right]. \tag{45}$$

The following theorem is useful:

Theorem 5 ($SU(2)$ disentangling theorem)**.** *Given $J_\pm$ and J_3 that obey the commutation relations*

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \tag{46}$$

$$\exp\left(\lambda_+ J_+ + \lambda_- J_- + \lambda_3 J_3\right) = \exp\left(\Lambda_+ J_+\right)\Lambda_3^{J_3}\exp\left(\Lambda_- J_-\right), \tag{47}$$

where

$$\Lambda_\pm \equiv \frac{2\lambda_\pm\sinh\xi}{2\xi\cosh\xi - \lambda_3\sinh\xi}, \qquad \Lambda_3 \equiv \left(\cosh\xi - \frac{\lambda_3}{2\xi}\sinh\xi\right)^{-2}, \qquad \xi^2 \equiv \frac{1}{4}\lambda_3^2 + \lambda_+\lambda_-. \tag{48}$$

Proof. See, for example, Ref. [23]. □

For the case of interest here,

$$\lambda_3 = i\phi(\nu-\mu), \quad \lambda_+ = \gamma\phi, \quad \lambda_- = -\gamma\phi, \tag{49}$$

$$\xi = \sqrt{-\frac{1}{4}\phi^2(\nu-\mu)^2 - \gamma^2\phi^2} = i\frac{1+\lambda}{2}\phi, \tag{50}$$

$$\Lambda_3 = \left[\cos\frac{1+\lambda}{2}\phi - i(1-2\eta)\sin\frac{1+\lambda}{2}\phi\right]^{-2}. \tag{51}$$

The disentangling theorem is useful because $\exp(\Lambda_- J_-)|0\rangle_B = |0\rangle_B$:

$$
\begin{aligned}
{}_B\langle 0|\exp(ig\phi)|0\rangle_B &= {}_B\langle 0|\exp\left[i\phi(\mu+\nu)J\right]\exp\left(\Lambda_+ J_+\right)\Lambda_3^{J_3}\exp\left(\Lambda_- J_-\right)|0\rangle_B && (52)\\
&= \exp\left[i\phi(\mu+\nu)a^\dagger a/2\right]\Lambda_3^{-a^\dagger a/2} && (53)\\
&= z^{a^\dagger a}, && (54)\\
z &\equiv \eta e^{i\phi} + (1-\eta)e^{-i\lambda\phi}. && (55)
\end{aligned}
$$

Generalization to the multimode case is straightforward. For continuous optical fields, $a(t)$ can be first discretized in time as $a(t_j) \approx \sqrt{\delta t}a_j$ before applying the multimode result and taking the continuous limit.

Quantum Fisher information

Consider the multimode case with annihilation operators a_j and b_j. Let

$$U_A = \exp\left(i\sum_j \phi_j a_j^\dagger a_j\right), \qquad U_{AB} = \exp\left[i\kappa\sum_j\left(a_j^\dagger b_j + a_j b_j^\dagger\right)\right], \qquad U_B = \exp\left(i\lambda\sum_j \phi_j b_j^\dagger b_j\right). \tag{56}$$

The QFI matrix $J^{(Q)}$ can be computed by considering the fidelity for small ϕ_j and $F \approx 1$ [20]:

$$F = \left|\langle\Psi|U_{AB}^\dagger U_B^\dagger U_A U_{AB}|\Psi\rangle\right|^2 \approx 1 - \frac{1}{4}\sum_{jk}\phi_j J_{jk}^{(Q)}\phi_k + O(||\phi||^4). \tag{57}$$

This also shows why F is more difficult to calculate than $J^{(Q)}$ in general, as $J^{(Q)}$ is just a second-order term in F. Writing the fidelity as

$$F \equiv \left|\langle\Psi|\exp\left(i\sum_j g_j\phi_j\right)|\Psi\rangle\right|^2, \tag{58}$$

$$g_j \equiv \mu a_j^\dagger a_j + \nu b_j^\dagger b_j + i\gamma(a_j^\dagger b_j - a_j b_j^\dagger), \tag{59}$$

I get

$$J_{jk}^{(Q)} = 4\langle\Psi|\Delta g_j\Delta g_k|\Psi\rangle, \qquad \Delta g_j \equiv g_j - \langle\Psi|g_j|\Psi\rangle. \tag{60}$$

After some algebra,

$$\frac{J_{jk}^{(Q)}}{4} = (\cos^2\kappa - \lambda\sin^2\kappa)^2\langle\Delta n_j\Delta n_k\rangle + (1+\lambda)^2\sin^2\kappa\cos^2\kappa\,\langle n_j\rangle\,\delta_{jk}, \tag{61}$$

where $n_j \equiv a_j^\dagger a_j$. In the continuous-time limit with $t_j = t_0 + j\delta t$ [4],

$$\frac{1}{\sqrt{\delta t}}a_j \to a(t_j), \qquad \frac{1}{\delta t}n_j \to I(t_j), \qquad \frac{\delta_{jk}}{\delta t} \to \delta(t_j - t_k), \qquad \frac{J_{jk}^{(Q)}}{\delta t^2} \to J^{(Q)}(t_j, t_k), \tag{62}$$

and Eq. (27) in the main text is obtained.